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Existence and Topological Uniqueness of Compact CMC Hypersurfaces with Boundary in Hyperbolic Space

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Abstract It is proved that if Γ is a compact, embedded hypersurface in a totally geodesic hypersurface $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$ satisfying the enclosing H-hypersphere condition with $|H| < 1$, then there is one and only one (up to a reflection on $\mathbb{H}^n$) compact embedded constant mean curvature H hypersurface M such that $\partial M = \Gamma$. Moreover, M is diffeomorphic to a ball.

Keywords Constant mean curvature · Hyperbolic space · Hyperbolic Killing graphs · Alexandrov reflection method

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1 Introduction

Let $\mathbb{H}^{n+1}$ be the hyperbolic space of constant sectional curvature -1. By a hypersphere of $\mathbb{H}^{n+1}$ we mean a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ whose mean curvature has absolute value strictly smaller than 1. Given $-1 < H < 1$, $p \in \mathbb{H}^{n+1}$, and a unit tangent vector $v \in T_p\mathbb{H}^{n+1}$, there is only one hypersphere $E^n_{p,v,H}$ of $\mathbb{H}^{n+1}$ passing through p which has mean curvature H with respect to the unit normal vector field η such that $\eta(p) = v$ (for more details, see the next section).

Let $\mathbb{H}^n$ be a totally geodesic hypersurface of $\mathbb{H}^{n+1}$, Ω a smooth domain in $\mathbb{H}^n$, and η the unit normal vector field along $\partial\Omega$ pointing to Ω. Given $0 \le H < 1$, we say that Ω satisfies the enclosing H-hypersphere condition if for any $p \in \partial\Omega$ the connected component of $\mathbb{H}^n \backslash E^{n-1}_{p,\eta(p),H}$ to which $\eta(p)$ is pointing contains Ω (this definition is a natural extension to the hyperbolic space of the enclosing sphere condition used in Euclidean PDE theory; see [6], p. 339).

A Killing vector field of $\mathbb{H}^{n+1}$ is called hyperbolic if its integral curves are hypercycles orthogonal to a totally geodesic hypersurface of $\mathbb{H}^{n+1}$. Given an oriented geodesic γ there is a unique hyperbolic Killing vector field X tangent to γ in the orientation of γ. Moreover, X is orthogonal to the totally geodesic hypersurfaces of $\mathbb{H}^{n+1}$ which are orthogonal to γ.

Let X be a hyperbolic Killing vector field of $\mathbb{H}^{n+1}$ orthogonal to a totally geodesic hypersurface $\mathbb{H}^n$ of $\mathbb{H}^{n+1}$ and denote by φ_t the one-parameter subgroup of isometries determined by X, $\varphi_0 = \mathrm{Id}_{\mathbb{H}^{n+1}}$. The X-Killing graph $\mathrm{Gr}(u)$ of a function u defined in a subset S of $\mathbb{H}^n$ is $\mathrm{Gr}(u) = \{\varphi_{u(x)}(x) | x \in S\}$. In the half-space model for $\mathbb{H}^{n+1}$, that is, $\mathbb{R}^{n+1}_+$ with the metric $dz^2 = (1/x^2_{n+1})dx^2$, where dx^2 is the Euclidean metric, if the geodesic γ is the oriented x_{n+1} axis, then $\varphi_t(x) = e^t x$, $x \in \mathbb{H}^{n+1}$, and the hyperbolic graphs are radial graphs over the totally geodesic half-sphere $x_1^2 + \cdots + x^2_{n+1} = 1$, $x_{n+1} > 0$. We prove:

Theorem 1.1 *Let $\mathbb{H}^n$ be a totally geodesic hypersurface of $\mathbb{H}^{n+1}$.*

(a) *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{H}^n$ satisfying the enclosing H-hypersphere condition, $0 \le H < 1$, and let γ be an oriented geodesic passing orthogonally through Ω and X the hyperbolic Killing field tangent to γ in the orientation of γ. Then there is a unique $u \in C^{2,\alpha}(\overline{\Omega})$ such that $u|_{\partial\Omega} = 0$ and the X-Killing graph of u, oriented with a normal vector field η such that $\langle \eta, X\rangle \le 0$, has constant mean curvature (CMC) H.*

(b) *Let M be a compact, embedded, CMC H hypersurface of $\mathbb{H}^{n+1}$ such that $\partial M \subset \mathbb{H}^n$ is the boundary of a domain $\Omega \subset \mathbb{H}^n$ satisfying the enclosing H-hypersphere condition, $0 \le H < 1$. Then M is a graph with respect to any hyperbolic Killing vector field tangent to a geodesic of $\mathbb{H}^{n+1}$ orthogonal to Ω. In particular, M is diffeomorphic to an n-dimensional ball.*

Existence and uniqueness of compact constant mean curvature hypersurfaces with boundary in an umbilical hypersurface of the hyperbolic space have been studied by many authors. When the boundary is contained in a totally geodesic hypersphere, it is known that if the mean curvature H_C of the hyperbolic cylinder C over $\partial\Omega$ (that is, $C = \{\varphi_t(x) | x \in \partial\Omega, t \in \mathbb{R}\}$) satisfies $H_C \geq H$, then there exists $u \in C^{2,\alpha}(\overline{\Omega})$ as stated in (a). This is a consequence of the more general Theorem 1.1 of [4] when $n = 2$ and Theorem 1 of [3] in arbitrary dimensions. In $\mathbb{H}^{n+1}$ the existence of CMC H hyperbolic Killing graphs has also been proved in [11] and in the recent work [5], both requiring the strict inequality $H_C > H$. We note that the H-enclosing hypersphere condition does not imply $H_C \geq H$. Existence of CMC H hypersurfaces with boundary in a horosphere (CMC 1 umbilical hypersurfaces) is proved in [9] (Theorem 1.1) and in [10] (Theorem 1.1) with hypotheses which are similar to ours.

Regarding result (b), we observe that it follows from a well-known theorem of A.D. Alexandrov [1] that an embedded compact CMC hypersurface M in a simply connected space form $\overline{M}^{n+1}$ is a totally umbilical round hypersphere (when the space form is a sphere, the hypersurface is required to be contained in a hemisphere). In the context of embedded compact CMC hypersurfaces with non-empty boundary, the following topological problem has been investigated by several mathematicians: Let $\Pi \subset \overline{M}^{n+1}$ be a totally geodesic hypersurface, and let M be an embedded compact CMC hypersurface with connected boundary $\partial M \subset \Pi$. Find natural geometric conditions under which M is a topological n-dimensional ball.

This problem was considered in [2] for surfaces in the Euclidean space $\mathbb{R}^3$, where the authors conjectured that a compact constant mean curvature surface in $\mathbb{R}^3$ bounded by a circle is a spherical cap if either the surface has genus 0 and it is immersed or the surface is embedded. As pointed out in [2], the conjecture holds for the subclass of surfaces that are embedded and contained in a half-space by observing that, in that case, the surface inherits the symmetries of its boundary. It is therefore of interest to obtain natural geometric conditions that force a compact embedded constant mean curvature surface M in $\mathbb{R}^3$ with planar boundary $\partial M \subset \Pi$ to be contained in one of the half-spaces of $\mathbb{R}^3$ determined by Π. In this respect, it was proved in [2] that this is true assuming additionally that ∂M is convex in Π and that M is transverse to Π along ∂M. We note that this problem can naturally be stated in $\overline{M}^{n+1}$ and implies a topological version of Alexandrov's theorem.

In Euclidean space $\mathbb{R}^{n+1}$ some progress has been made. Denoting by $\Omega \subset \Pi$ the domain enclosed by ∂M, it is proved in [7] that if M is locally a graph around ∂M (with no assumption on the convexity), then M is globally a graph on Ω, showing that M is a topological n-dimensional ball. When $n = 2$, in [9] it is proved that there exists a number $V_0 > 0$ depending only on ∂M such that if the volume V of the surface satisfies $|V| \leq V_0$, then M is a graph on Ω. In [12] it is proved that if $H \leq (\min \kappa)(\min \sqrt{1 - (\kappa_g/\kappa)^2})$, where κ is the planar curvature of ∂M and κ_g is the geodesic curvature of ∂M in M, then M is a round cap sphere. Although in all these results the hypotheses depend on the hypersurface M, one can expect that the topology of M is essentially determined by H and ∂M. In fact, under the assumption that M is contained in a half-space of $\mathbb{R}^3$, it is proved in [13] that there exists a constant $C(\kappa) > 0$ depending only on the curvature κ of ∂M such that if $0 \leq H \leq C(\kappa)$ then the surface is a topological disk. An explicit expression of $C(\kappa)$ has not been found so far. Theorem 3.3 of [4] also shows that the topology of M is determined only by the geometry of ∂M and H. Our result improves Theorem 3.3 of [4] since it replaces the enclosing *sphere* condition required in item (i) of Theorem 3.3 by the weaker enclosing *hypersphere* condition. We point out that there is a significant difference between these hypotheses: Under the enclosing sphere condition it easily follows that M is contained in the hyperbolic cone over ∂M and the result is then an immediate application either of Theorem 1.1 of [4] or of item (a) of our theorem.

2 Preliminaries

We shall make use of the following basic facts.

Lemma 2.1 *Let $\mathbb{H}^n$ be a totally geodesic hypersurface of $\mathbb{H}^{n+1}$. Given $p \in \mathbb{H}^n$, $v \in T_p\mathbb{H}^n$, $|v| = 1$, and $0 < |H| < 1$, we have $E^{n-1}_{p,v,H} = E^n_{p,v,H} \cap \mathbb{H}^n$.*

Lemma 2.2 *Let E be an H-hypersphere in $\mathbb{H}^{n+1}$, $H \neq 0$, and o be a point of the connected component of $\mathbb{H}^{n+1} \backslash E$ towards which $\overrightarrow{H}$ is pointing to, where $\overrightarrow{H}$ denotes the mean curvature vector field of E. Let $p \in E$ and ν be the exterior unit normal vector to a geodesic sphere centered at o passing through p. Then $\langle \nu, \overrightarrow{H}(p) \rangle < 0$.*

For a proof of Lemma 2.1 and Lemma 2.2, it will be appropriate for us to use the Minkowskian model of the hyperbolic space. Write $\mathbb{R}^{n+2}_1$ for $\mathbb{R}^{n+2}$, with canonical coordinates $(x_0, x_1, \ldots, x_{n+1})$, endowed with the Lorentzian metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.$$

The $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is the complete simply connected Riemannian manifold with sectional curvature -1, which is realized as the hyperboloid

$$\mathbb{H}^{n+1} = \left\{ x \in \mathbb{R}^{n+2}_1 : \langle x, x \rangle = -1, x_0 > 0 \right\} \subset \mathbb{R}^{n+2}_1$$

endowed with the Riemannian metric induced from $\mathbb{R}^{n+2}_1$. In this model, the H-hyperspheres are given by

$$\Sigma^n(a, \tau) = \left\{ x \in \mathbb{H}^{n+1} : \langle a, x \rangle = \tau \right\},$$

where $a \in \mathbb{R}^{n+2}_1$ satisfies $\langle a, a \rangle = 1$ and $\tau \neq 0$. It is not difficult to see that the mean curvature vector field of $\Sigma^n(a, \tau)$ is given by

$$\overrightarrow{H}_{a,\tau}(x) = \frac{-\tau}{1+\tau^2}(a + \tau x)$$

for every $x \in \Sigma^n(a, \tau)$. Therefore, given $0 < |H| < 1$, $p \in \mathbb{H}^{n+1}$, and a unit tangent vector $v \in T_p\mathbb{H}^{n+1}$, the only H-hypersphere $E^n_{p,v,H}$ passing through p and having mean curvature H with respect to the unit normal field η such that $\eta(p) = v$ is the H-hypersphere $\Sigma^n(a, \tau)$ with

$$a = -\frac{H}{\sqrt{1-H^2}} p - \frac{1}{\sqrt{1-H^2}} v, \quad \text{and} \quad \tau = \frac{H}{\sqrt{1-H^2}}.$$

This implies that if $p \in \mathbb{H}^n \subset \mathbb{H}^{n+1}$ and $v \in T_p\mathbb{H}^n$, then

$$E^{n-1}_{p,v,H} = \Sigma^{n-1}(a,\tau) = \Sigma^n(a,\tau) \cap \mathbb{H}^n = E^n_{p,v,H} \cap \mathbb{H}^n.$$

This proves Lemma 2.1.

On the other hand, let E be an H-hypersphere in $\mathbb{H}^{n+1}$, $H \neq 0$. Without loss of generality, we may assume that

$$E = \Sigma^n(a,\tau) = \left\{x \in \mathbb{H}^{n+1} : \langle a, x\rangle = \tau\right\},$$

where $a \in \mathbb{R}^{n+2}_1$ satisfies $\langle a, a\rangle = 1$ and $\tau > 0$ (otherwise, replace a by $-a$). Denote by E^+ the connected component of $\mathbb{H}^{n+1} \backslash E$ towards which $\overrightarrow{H}$ is pointing to, where $\overrightarrow{H}$ denotes the mean curvature vector field of E, that is,

$$\overrightarrow{H}(x) = \frac{-\tau}{1+\tau^2}(a + \tau x)$$

for every $x \in E$. We claim that $E^+ = \{x \in \mathbb{H}^{n+1} : \langle a, x\rangle < \tau\}$. To see this, take $x \in E$ and let

$$\gamma(t) = \cosh(t)x + \sinh(t)\overrightarrow{H}(x)$$

be the geodesic starting at x with velocity $\overrightarrow{H}(x)$. It is clear that

$$\left\langle a, \gamma(t)\right\rangle = \tau e^{-t} < \tau$$

for every $t > 0$. Thus, $E^+ = \{x \in \mathbb{H}^{n+1} : \langle a, x\rangle < \tau\}$ as claimed. Choose a point $o \in E^+$ and let $p \in E$. Define $s(\cdot) = d(o, \cdot)$, where d is the Riemannian distance in $\mathbb{H}^{n+1}$. As is well known, the exterior unit normal vector to the geodesic sphere centered at o and passing through p is given by

$$\nu = \operatorname{grad} s(p).$$

Recall that $s(\cdot) = d(o, \cdot) = \operatorname{arg\,cosh}(-\langle o, \cdot\rangle)$. In particular, for every $v \in T_p\mathbb{H}^{n+1}$,

$$v(s) = \left\langle \operatorname{grad} s(p), v\right\rangle = \frac{-\langle o, v\rangle}{\sinh(s(p))}.$$

Thus,

$$\begin{aligned}\left\langle \nu, \overrightarrow{H}(p)\right\rangle &= \frac{-\tau}{1+\tau^2}\left\langle \operatorname{grad} s(p), a + \tau p\right\rangle = \frac{\tau\langle o, a+\tau p\rangle}{(1+\tau^2)\sinh(s(p))} \\ &= \frac{\tau(\langle o, a\rangle + \tau\langle o, p\rangle)}{(1+\tau^2)\sinh(s(p))} < \frac{\tau^2(1+\langle o, p\rangle)}{(1+\tau^2)\sinh(s(p))} < 0,\end{aligned}$$

where we have used the facts that $\langle o, a\rangle < \tau$ and $\langle o, p\rangle < -1$. This completes the proof of Lemma 2.2.

3 Proof of the Theorem

Proof of (a) Assume that the geodesic γ passes through the point $o \in \Omega$ and let X be the hyperbolic Killing field tangent to γ. Let $\mathbb{H}_-^{n+1}$ stand for the connected component of $\mathbb{H}^{n+1} \setminus \mathbb{H}^n$ to which $X(o)$ points.

From an n-dimensional version of Proposition 2.1 of [4], we have to prove the existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the Dirichlet problem

$$\begin{cases} Q_H(u) := \operatorname{div} \frac{\rho \operatorname{grad} u}{\sqrt{1+\rho^2 |\operatorname{grad} u|^2}} + \frac{\langle \operatorname{grad} u, \operatorname{grad} \rho \rangle}{\sqrt{1+\rho^2 |\operatorname{grad} u|^2}} + nH = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $\rho(x) = \|X(x)\|^2$, $x \in \mathbb{H}^{n+1}$. We use the standard open closed argument to prove that the set

$$V = \left\{ t \in [0, 1];\ \exists u \in C^{2,\alpha}(\overline{\Omega}) \text{ such that } Q_{tH}(u) = 0,\ u|_{\partial\Omega} = 0 \right\}$$

is $[0, 1]$. Clearly $0 \in V$ so that $V \neq \emptyset$ and, by the Implicit Function Theorem, V is open. Choose a $t \in [0, 1]$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ satisfy $Q_{tH}(u) = 0$ and $u|_{\partial\Omega} = 0$. The function $v = 0$ is a subsolution since $Q_{tH}(v) = tnH \geq 0$. It follows that $u \geq 0$.

Let E be an H-hypersphere contained in $\mathbb{H}_-^{n+1}$ orthogonal to the geodesic γ and oriented by a normal vector N_E such that $\langle N_E, X \rangle < 0$. Then E is an X-Killing graph of a strictly positive function z and $Q_H(z) = 0$. It follows that z is a supersolution for Q_{tH} since $Q_{tH}(z) = -nH + tnH = nH(t-1) \leq 0$. Since $z|_{\partial\Omega} > 0$ we have $u \leq z$. We then have the C^0 a priori estimate

$$|u|_0 \leq C = \max_{\overline{\Omega}} z. \tag{2}$$

To obtain C^1 estimates, we will construct local barriers from above at any point of $\partial\Omega$ with uniform C^1 bounds. Precisely, we will prove the existence of a constant D such that, given $p \in \partial\Omega$, there is a $C^{2,\alpha}$ neighborhood U_p of p in $\overline{\Omega}$, and a function $w_p \in C^{2,\alpha}(\overline{U}_p)$ satisfying the following properties:

(i) $Q_{tH}(w_p) \leq 0$ for every $t \in [0, 1]$,
(ii) $w_p|_{U_p} \geq u|_{U_p}$ for every solution $u \in C^{2,\alpha}(\overline{\Omega})$ of $Q_{tH}(u) = 0$ such that $u|_{\partial\Omega} = 0$, and
(iii) $\max_{U_p} |\operatorname{grad} w_p| \leq D$.

If that is the case, since $0 \leq u \leq w_p$, it follows that $\max_{\partial\Omega} |\operatorname{grad} u| \leq D$. Therefore, from Lemma 11 of [3] and the C^0 estimate (2) we have a priori C^1 estimates of any solution of $u \in C^{2,\alpha}(\overline{\Omega})$ of $Q_{tH}(u) = 0$ such that $u|_{\partial\Omega} = 0$. From PDE elliptic theory we have $V = [0, 1]$.

To show the existence of these local barriers, since a barrier from above for Q_H is also a barrier from above for Q_{tH}, we may assume that $t = 1$. Let E_p^{n-1} be the hypersphere in $\mathbb{H}^n$ through p given by the interior H-hypersphere condition. Define $r(x) = d(x, E_p^{n-1})$, $x \in \Omega$, where d is the Riemannian distance in $\mathbb{H}^n$, and let $w(x) := w_p(x) = f(r(x))$ for a certain $f \in C^2(\mathbb{R})$ satisfying $f \geq 0$ and $f(0) = 0$

given in the sequel. Noting that $\operatorname{grad} w = f' \operatorname{grad} r$, we obtain, after a computation,

$$\begin{aligned}\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w) &= \left(1+\rho^2 f'^2\right)\left(\rho f' \Delta r + nH\sqrt{1+\rho^2 f'^2}\right)\\ &\quad + \left(2+\rho^2 f'^2\right) f' \langle \operatorname{grad} r, \operatorname{grad} \rho\rangle + \rho f''.\end{aligned}$$

We choose $f(r)$ of the form $f(r) = L\ln(1+K^2 r)$ where L and K are constants to be determined later. The function f satisfies $f' > 0$ and $f'' = -f'^2/L$. Since $\sqrt{1+\rho^2 f'^2} \le 1+\rho f'$, we have

$$\rho f' \Delta r + nH\sqrt{1+\rho^2 f'^2} \le \rho f'(\Delta r + nH) + nH.$$

Then

$$\begin{aligned}\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w) &\le \left(1+\rho^2 f'^2\right)\left(\rho f'(\Delta r + nH) + nH\right)\\ &\quad + \left(2+\rho^2 f'^2\right) f' \langle \operatorname{grad} r, \operatorname{grad} \rho\rangle - \frac{\rho f'^2}{L}.\end{aligned} \tag{3}$$

We now show that the function $\langle \operatorname{grad} r, \operatorname{grad} \rho\rangle$ is negative at p. We define the function $s(x) = d(x, o)$. Since ρ is radially symmetric with respect to o, $\operatorname{grad}\rho$ is orthogonal to the geodesic spheres centered at o. Then $\operatorname{grad}\rho$ is proportional to $\operatorname{grad} s$ at p. Since X is a Killing vector field, we have, at the point $x = p$,

$$\begin{aligned}\langle \operatorname{grad} s, \operatorname{grad}\rho\rangle &= \operatorname{grad} s\left(\langle X, X\rangle\right) = 2\langle \nabla_{\operatorname{grad} s} X, X\rangle\\ &= -2\langle \nabla_X X, \operatorname{grad} s\rangle = -2\big\langle \alpha(X, X), \operatorname{grad} s\big\rangle > 0,\end{aligned}$$

where α is the second fundamental form of the cone around the geodesic γ. It follows that $\operatorname{grad}\rho(p) = |\operatorname{grad}\rho(p)| \operatorname{grad} s(p)$. Moreover, since $\overrightarrow{H}(p) = H \operatorname{grad} r(p)$, where $\overrightarrow{H}(p)$ is the mean curvature vector of E_p^{n-1} at p, we obtain, using Lemma 2.2,

$$\langle \operatorname{grad} r, \operatorname{grad}\rho\rangle(p) = (1/H)|\operatorname{grad}\rho|\big\langle \operatorname{grad} s, \overrightarrow{H}\big\rangle(p) < 0.$$

Thus there exists $r_1 > 0$ such that $U_{r_1} = \{x \in \overline{\Omega}; r(x) \le r_1\}$ is a neighborhood of p where $\langle \operatorname{grad} r, \operatorname{grad}\rho\rangle(x) < 0$ for all $x \in U_{r_1}$. Moreover, observing that $\Delta r = -nH_r$ where H_r is the mean curvature of the umbilical hypersurface at a distance r of E_p and contained in the connected component of $\mathbb{H}^n$ where the mean curvature vector of E_p is pointing to, we obtain from (3):

$$\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w)|_{U_{r_1}} \le \left(1+\rho^2 f'^2\right)\left(n\rho f'(H - H_r) + nH\right) - \frac{\rho f'^2}{L}.$$

Since $H_r \to H$ as $r \to 0$, letting $r \to 0$ we obtain

$$\begin{aligned}\lim_{r(x)\to 0}\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w)(x) &\le nH\left(1+\rho^2(p)\left(f'(0)\right)^2\right) - \frac{\rho(p)(f'(0))^2}{L}\\ &= \rho(p)K^4 L\left(nH\rho(p)L - 1\right) + Hn.\end{aligned}$$

Choosing $L = 2C/\ln(1+K)$, where C is given by (2), we obtain

$$\lim_{r(x)\to 0}\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w)(x) \le \frac{2C\rho(p)K^4}{\ln(1+K)}\left(\frac{2CnH\rho(p)}{\ln(1+K)}-1\right)+Hn.$$

Since $\rho(p)>0$, then

$$\lim_{K\to+\infty}\frac{2C\rho(p)K^4}{\ln(1+K)}\left(\frac{2CnH\rho(p)}{\ln(1+K)}-1\right)+Hn=-\infty,$$

and we may choose K_0 sufficiently large, and depending only on p, n, C, and H, such that (say)

$$\frac{2C\rho(p)K^4}{\ln(1+K)}\left(\frac{2CnH\rho(p)}{\ln(1+K)}-1\right)+Hn \le -2$$

for every $K \ge K_0$. Therefore, we may then choose a positive number $K_1 > K_0$ sufficiently large such that $1/K_1 \le r_1$ and if $r(x) \le 1/K_1$, then

$$\left(1+\rho^2 f'^2\right)^{3/2} Q_H(w)(x) < 0.$$

This means that w is a supersolution for Q_H in U_{1/K_1}. Moreover, since $w|_{\partial U_{1/K_1}\cap\partial\Omega} = 0 = u|_{\partial\Omega}$ and

$$w|_{\partial U_{1/K_1}\setminus\partial\Omega} = f(1/K_1) = 2C > u|_{\partial U_{1/K_1}\setminus\partial\Omega},$$

the function w is a local upper barrier for problem (1) in a neighborhood of p. By comparison, we obtain the a priori bound

$$|\operatorname{grad} u|(p) \le |\operatorname{grad} w|(p).$$

Since $D := \inf_\Omega \rho > 0$ it is clear that one may choose an a priori gradient estimate of a solution of (1) that depends only on n, H, C, and D. This guarantees the existence of a graph G and completes the proof of item (a). The uniqueness is an immediate consequence of the maximum principle. □

Proof of (b) The idea of the proof is as follows. From the enclosing H-hypersphere condition and the tangency principle it follows that the hypersurface does not intersect $\mathbb{H}^n \setminus \overline{\Omega}$. As a consequence of comparing M with the family of totally geodesic hyperplanes obtained by moving $\mathbb{H}^n$ through a one-parameter subgroup of isometries of $\mathbb{H}^{n+1}$, we show that M lies in a connected component $\mathbb{H}^{n+1}_+$ of $\mathbb{H}^{n+1}\setminus\mathbb{H}^n$. Now we may use item (a) to assert the existence of a hyperbolic Killing graph G on Ω, with respect to a fixed but arbitrary geodesic passing orthogonally through Ω, with $\partial M = \partial G$ and contained in $\mathbb{H}^{n+1}\setminus\mathbb{H}^{n+1}_+$. Thus $M\cup G$ defines an embedded closed hypersurface which may be singular at $\partial\Omega$; however, from the boundary tangency principle and the enclosing hypersphere condition the tangent spaces of M and G along $\partial\Omega$ have an inner angle strictly smaller than π. We then use the Alexandrov reflection technique on $M\cup G$ and prove that M must be a Killing graph. Let us now develop in detail this sketch.

The case $H=0$ in the theorem is immediate: the tangency principle implies that $M=\Omega$ and the theorem is trivial in this case. We may then assume that $H>0$. Since $M\cup\Omega$ is a topological immersed hypersurface without boundary, it divides $\mathbb{H}^{n+1}$ into connected components, one of them, say U, being unbounded. It is also an immediate consequence of the tangency principle that the mean curvature vector of M points to $\mathbb{H}^{n+1}\backslash U$. Denote by $\mathbb{H}^{n+1}_+$ the closure of the connected component of $\mathbb{H}^{n+1}\backslash\mathbb{H}^n$ that contains points of $M\backslash\partial M$. □

Claim 1 *It holds that* $M\cap(\mathbb{H}^{n+1}\backslash\mathbb{H}^{n+1}_+)=\emptyset$.

Proof of Claim 1 Let η be the unit normal vector field along $\partial\Omega$ pointing to Ω. Let $p\in\partial\Omega$ be given and set $E^n_p=E^n_{p,\eta(p),H}$. We claim that the hypersurface M is contained in the closure $E^{n,+}_p$ of the connected component of $\mathbb{H}^{n+1}\backslash E^n_p$ which $\eta(p)$ is pointing to. In fact, first observe that, since Ω satisfies the enclosing H-hypersphere condition, it follows from Lemma 2.1 that $\Omega\subset E^{n,+}_p$. Now, let ψ_t be the one-parameter subgroup of isometries of $\mathbb{H}^{n+1}$ generated by the hyperbolic Killing field X_p which integral curves are hypercycles equidistant to a geodesic orthogonal to E^n_p, $\psi_0=\mathrm{Id}_{\mathbb{H}^{n+1}}$. Assume that $X(p)$ points to $E^{n,-}_p:=\mathbb{H}^{n+1}\backslash E^{n,+}_p$. Setting

$$A=\left\{t\geq 0;\ \psi_s\left(E^n_p\right)\cap M\neq\emptyset \text{ for all } s\in[0,t]\right\},$$

we have $A\neq\emptyset$ since $0\in A$. Because M is compact, $t_0:=\sup(A)<\infty$. We assert that $t_0=0$. By contradiction, assume that $t_0>0$. Then $\psi_{t_0}(E^n_p)$ is tangent to M and, moreover, M is contained in $\psi_{t_0}(E^{n,+}_p)$. As the mean curvature vectors of both surfaces agree at the tangent point, the tangency principle gives a contradiction.

We then have $t_0=0$ and $M\cap E^{n,-}_p=\emptyset$. Because this holds for any $p\in\partial\Omega$ the surface M does not intersect $\mathbb{H}^n\backslash\Omega$. Now it is enough to apply Theorem 2.2 of [8]. This completes the proof of Claim 1. □

Choose a point $o\in\Omega$ and let X be the hyperbolic Killing vector field which integral curves are hypercycles equidistant to the geodesic γ through o and orthogonal to $\mathbb{H}^n$. As in the proof of item (a), let $\mathbb{H}^{n+1}_-$ stand for the connected component of $\mathbb{H}^{n+1}\setminus\mathbb{H}^n$ to where $X(o)$ points, and assume that X induces the same orientation on γ. It then follows from item (a) the existence of a hyperbolic X-Killing graph G contained in the closure of $\mathbb{H}^{n+1}_-$ with CMC H with respect to the unit normal vector η such that $\langle\eta,X\rangle\leq 0$, and satisfying $\partial G=\partial\Omega$. Let φ_t the one-parameter subgroup of isometries determined by X, $\varphi_0=\mathrm{Id}$.

We have that $N:=M\cup G$ is a topological compact hypersurface without boundary which is not necessarily smooth along $\partial\Omega$ and has CMC H with respect to the inner orientation on $N\backslash\partial\Omega$. Denote by $W\subset\mathbb{H}^{n+1}$ the domain bounded by N. We now use the well-known Alexandrov technique by taking reflections with respect to the totally geodesic hypersurfaces $\varphi_t(\mathbb{H}^n)$. In this process, we will use the notation M_{t^+}, M_{t^-}, and $M^*_{t^+}$ as in [14], which we recall here for the reader's convenience. Recall that $\partial M\subset\varphi_0(\mathbb{H}^n)$. Then for every $t\in\mathbb{R}$, M_{t^+} denotes the portion of M which is on and above $\varphi_t(\mathbb{H}^n)$. Similarly, M_{t^-} denotes the portion of M which is on and below $\varphi_t(\mathbb{H}^n)$, while $M^*_{t^+}$ denotes the reflection of M_{t^+} across $\varphi_t(\mathbb{H}^n)$.

Starting with reflections with $t > 0$, we arrive until the time that $\varphi_t(\mathbb{H}^n)$ intersects M for the first time. After that, we begin considering t such that $M_{t^+} \subset W$ and denote

$$t_0 = \sup\{t;\ M^*_{s^+} \subset W, \text{for all } s \geq t\}.$$

Claim 2 *It holds that* $t_0 = 0$.

Proof of Claim 2 By contradiction, assume that $t_0 > 0$. Then one of the following three possibilities holds:

(1) $M^*_{t_0^+} \cap \partial\Omega \neq \emptyset$.
(2) There is a tangency between $M^*_{t_0^+}$ and G at an interior point of both $M^*_{t_0^+}$ and G.
(3) There is a boundary tangency between $M^*_{t_0^+}$ and $M_{t_0^-}$.

In the first case, since both M and G are contained in $E_p^{n,+}$ with $p \in M^*_{t_0^+} \cap \partial\Omega$ and they cannot be tangent to E_p^n at p, the inner angle between the tangent planes T_pM and T_pG at p is strictly smaller than π. Since $\partial M^*_{t_0^+} \cap \partial\Omega = \emptyset$ because $t_0 > 0$, then $M^*_{t_0^+}$ has to intersect $\partial\Omega$ at a point belonging to $M^*_{t_0^+} \backslash \partial M^*_{t_0}$, which is not possible since $M^*_{t_0^+}$ is smooth and totally contained in the interior of N. Observe that, by regularity, it is not possible for a smooth hypersurface to be at the same time entirely contained in the interior of N and to touch a point of $\partial\Omega$.

The second possibility cannot hold. Otherwise N would be a smooth surface, which is not possible since it is not smooth at $\partial\Omega$ as previously observed.

Finally, the third case is not possible either, since otherwise $N_{t_0} := M^*_{t_0^+} \cup M_{t_0^-}$ would be a compact embedded CMC surface (without boundary) and therefore, by Alexandrov's theorem, it would be a geodesic sphere with mean curvature bigger than 1, which is a contradiction. This completes the proof of Claim 2. ☐

We then have that $t_0 = 0$, and this proves that M is an X-hyperbolic Killing graph. Since $o \in \Omega$ is arbitrary, the proof of the theorem is finished.

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